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ELASTICITY PROBLEMS INVOLVING COUPLED HALF-PLANES†

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A general method is proposed for reducing problems concerning cracks, cuts, inclusions and interacting blocks in coupled half-planes to complex integral equations, both singular and hyper-singular. The method is based on the fact that if the Kolosov–Muskhelishvili functions are known for a whole plane, then the corresponding functions for coupled half-planes are obtained from them by simple transformations. Boundary integral equations (BIE) are presented, as well as fundamental solutions for isolated forces and periodic systems of forces, which may be used to construct new complex BIEs. © 2000 Elsevier Science Ltd. All rights reserved.

Problems concerning cracks, cuts and/or inclusions near the interface of media with different properties, in particular, near the free or attached boundary of a body, are of much interest in the study of materials, fracture mechanics and mining geomechanics. A great many references to publications treating special cases of this kind, primarily in relation to cracks, may be found, e.g. in [1–4]. On the other hand, as yet, general equations suitable for arbitrary contours of cracks, cuts and inclusions have been constructed only for the case of a half-plane with a free boundary [5] (see also [6]). In Section 2 below such BIEs will be derived for *arbitrary* coupled half-planes. This is achieved by direct application of a device presented in Section 1 to the previously obtained equations for a whole plane [7–10].

An alternative approach to constructing BIEs is the use of fundamental solutions. To use this approach for coupled half-planes one needs suitable solutions. Once again, such solutions have been constructed only for the case of a half-plane [10–12]. In Section 3 below, again using the device described in Section 1, we present the first construction of fundamental solutions for *arbitrary* coupled half-planes.

1. GENERAL FORMULAE FOR COUPLED HALF-PLANES

Consider two elastic half-planes S_1 and S_2 coupled along a straight boundary. In the general case, the shear moduli and Poisson's ratios are different: μ_1, ν_1 in the lower half-plane and μ_2, ν_2 in the upper half-plane. The position and configuration of cracks, cuts and/or inclusions in the half-planes may also differ. In the case of inclusions, each may have different elastic characteristics. Let L_1 denote the total contour of the system of cracks, cuts and/or inclusions in the lower half-plane, and let L_2 denote the total contour in the upper half-plane; $L = L_1 + L_2$ is the total contour. In the special case when $\mu_2 = 0$ we have only the lower half-plane with stress-free boundary. If $\mu_2 = \infty$, the problem corresponds to the case in which the boundary of the lower half-plane is rigidly attached. The interpretation of the cases $\mu_1 = 0$ and $\mu_1 = \infty$ is analogous. To simplify the discussion, we will assume that the individual contours comprising L_1 and L_2 are loaded in such a way that the total principal vector acting on each of them is equal to zero. A more general case is obtained by including special terms making allowance for the many-valuedness of the functions (see [1] for closed contours, [13] for cuts). Further simplification, without significant loss of generality, will be achieved by assuming in non-periodic problems that the stresses vanish at infinity.

We introduce a global system of coordinates xOy with the x axis pointing right along the common boundary of the half-planes and the y axis pointing upwards. The solution will be sought in complex form [1].

Let us assume that we have expressions for the Kolosov–Muskhelishvili functions (KM functions) in the case when the contour L_1 (or L_2) lies in the whole plane. Let $\varphi_1(z)$, $\psi_1(z)$, $\varphi'_1(z) = \Phi_1(z)$, $\psi'_1(z) = \Psi_1(z)$ denote the KM functions for the contour L_1 of the same type as those stipulated on L_1 in the original problem; similarly, let $\varphi_2(z)$, $\psi_2(z)$, $\varphi'_2(z) = \Phi_2(z)$, $\psi'_2(z) = \Psi_2(z)$ denote the KM functions for the problem on the contour L_2 in the whole plane with the properties of the upper half-plane. Under

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the conditions imposed above on the principal vectors of the forces on each of the contours and at infinity, functions with subscript 1 are holomorphic everywhere outside L_1, in particular, in the whole upper half-plane. Functions with subscript 2 are holomorphic outside L_2, in particular, in the whole lower half-plane. Al these functions vanish at infinity. The boundary values corresponding to these functions, of the stresses $\sigma_{f2}(t)$ and displacements $u_{f2}(t)$ on L_2, are assumed for the moment to be arbitrary; we will call them *fictitious* stresses and displacements, as indicated by the subscript f. Finally, these values may be given in special cases by integral representations of the KM potentials with known density.

Let σ denote the complex vector of stresses arising on an element of unit area with normal **n** pointing to the right of the direction **t** of passage through the element: $\sigma(z) = \sigma_{nn} + i\sigma_{nt}$. This vector will always be considered in the local system (**n**, **t**). Unlike σ, the vector of displacements u will be used only in global coordinates: $u(z) = u_x + iu_y$.

The KM functions with subscript 1 generate fields of stresses $\sigma_{f1}(t)$ and displacements $u_{f1}(t)$; the functions with subscript 2 generate stresses $\sigma_{f2}(t)$ and displacements $u_{f2}(t)$. By the standard formulae of complex representation [1], we have

$$\sigma_{fj}(z) = \Phi_j(z) + \overline{\Phi_j(z)} + [\overline{z\Phi_j'(z)} + \overline{\Psi_j(z)}]\frac{d\bar{z}}{dz}$$
$$2\mu_k\frac{du_{fj}}{dz} = \chi_k\Phi_j(z) - \overline{\Phi_j(z)} - [\overline{z\Phi_j'(z)} + \overline{\Psi_j(z)}]\frac{d\bar{z}}{dz} \tag{1.1}$$

where $k = 1$ if the point lies in the lower half-plane, $k = 2$ if the point lies in the upper half-plane, χ_k is the Muskhelishvili parameter: $\chi_k = 3 - 4\nu_k$ for plane deformation and $\chi_k = (3 - \nu_k)/(1 + \nu_k)$ for a plane stressed state; $j = 1, 2$.

We now introduce two important functions which, in the final analysis, will serve for the solution of problems involving coupled half-planes. We note, first of all, that the formulae for the stresses in (1.1) do not contain the elasticity constants. Hence the stresses $\sigma_{f1}(z)$ and $\sigma_{f2}(z)$ will remain continuous across the boundary $y = 0$ of the half-planes. Consequently, the sum $\sigma_{f1}(z) + \sigma_{f2}(z)$ will also be continuous.

Unlike the stresses, the displacements $u_{f1}(z)$ and $u_{f2}(z)$, in the general case when $\mu_1 \neq \mu_2 = \nu_1 \neq \nu_2$, experience a discontinuity at the boundary. Accordingly, the sum of the displacements $u_{f1}(z) + u_{f2}(z)$ will also experience a discontinuity. Letting a superscript plus (minus) denote limit values in the lower (upper) half-plane, we infer from (1.1) the following expression for the derivative of the jump of the sum at the interface $y = 0$

$$\Delta u_f'(x) = (u_{f1}'^{+} - u_{f1}'^{-}) + (u_{f2}'^{+} - u_{f2}'^{-}) =$$
$$= \left(\frac{\chi_1}{2\mu_1} - \frac{\chi_2}{2\mu_2}\right)[\Phi_1(x) + \Phi_2(x)] + \left(\frac{1}{2\mu_2} - \frac{1}{2\mu_1}\right)[\Phi_{01}(x) + \Phi_{02}(x)] \tag{1.2}$$
$$\Phi_{01}(x) = \overline{\Phi}_1(x) + x\overline{\Phi_1'}(x) + \overline{\Psi}_1(x), \quad \Phi_{02}(x) = \overline{\Phi}_2(x) + x\overline{\Phi_2'}(x) + \overline{\Psi}_2(x)$$

where the prime in this case denotes differentiation in the direction of the x axis, and $\Phi_{01}(x)$ and $\Phi_{02}(x)$ are the limit values of the functions

$$\Phi_{01}(z) = \overline{\Phi}_1(z) + z\overline{\Phi_1'}(z) + \overline{\Psi}_1(z), \quad \Phi_{02}(z) = \overline{\Phi}_2(z) + z\overline{\Phi_2'}(z) + \overline{\Psi}_2(z) \tag{1.3}$$

which are of key importance for the following constructions—with $\bar{g}(z) = \overline{g(\bar{z})}$ by definition for any function $g(z)$. It is important that $\Phi_{01}(x)$ is holomorphic in the lower half-plane and $\Phi_{02}(x)$ in the upper half-plane.

It follows from the above that a solution of the original problem will be found if we construct two pairs of KM functions $\Phi_{a1}(z)$, $\Psi_{a1}(z)$ and $\Phi_{a2}(z)$, $\Psi_{a2}(z)$, holomorphic in each of the half-planes, discontinuous along the x axis, and such that the stresses $\sigma_{a1}(z)$ in the lower half-plane corresponding to $\Phi_{a1}(z)$, $\Psi_{a1}(z)$ transfer continuously into the stresses $\sigma_{a2}(z)$ in the upper half-plane corresponding to $\Phi_{a2}(z)$, $\Psi_{a2}(z)$, whereas the difference $u_{a1}(z) - u_{a2}(z)$ of the corresponding displacements compensates for the jump $\Delta u'_f(x)$ at the boundary of the half-planes: $\Delta u'_{a1}(x) - u'_{a2}(x) = -\Delta'_f(x)$. Then, by the KM formulae, the sums

$$\Phi(z) = \Phi_1(z) + \Phi_2(z) + \Phi_{aj}(z), \quad \Psi(z) = \Psi_1(z) + \Psi_2(z) + \Psi_{aj}(z) \tag{1.4}$$

where $j = 1$ in the lower half-plane and $j = 2$ in the upper half-plane, give the total stresses and displacements, which are continuous at the interface of the media whatever the fictitious stresses $\sigma_f(t)$ and displacements $u_f(t)$. In sum, it remains to choose the fictitious values so that the total stresses and/or displacements satisfy the given boundary conditions on L_1 and L_2. This at once leads to complex equations for the fictitious quantities. Thus, the problem has been reduced to determining the additional functions $\Phi_{aj}(z)$ and $\Psi_{aj}(z)$ in each of the half-planes.

The additional functions $\Phi_{aj}(z)$ and $\Psi_{aj}(z)$ ($j = 1, 2$) are quite easy to determine. Indeed, as we know [1, Section 112], taking $\Psi_{aj}(z)$ in the form

$$\Psi_{aj}(z) = -[\Phi_{aj}(z) + \overline{\Phi_{aj}}(z) + z\Phi'_{aj}(z)] \tag{1.5}$$

we obtain additional stresses

$$\sigma_{aj}(z) = \Phi_{aj}(z) - \Phi_{aj}(\bar{z})\frac{d\bar{z}}{dz} + \left(1 - \frac{d\bar{z}}{dz}\right)\overline{\Phi_{aj}(z)} + (z - \bar{z})\overline{\Phi'_{aj}(z)}\frac{d\bar{z}}{dz} \tag{1.6}$$

which have the following simple expressions on the interface $y = 0$ ($z = \bar{z} = x$, $d\bar{z}/dz = 1$).

$$\sigma^+_{a1}(x) = \Phi^+_{a1}(x) - \Phi^-_{a1}(x), \quad \sigma^-_{a2}(x) = \Phi^-_{a2}(x) - \Phi^+_{a2}(x)$$

Consequently, when representation (1.5) is used, continuity of the stresses implies that necessarily $(\Phi_{a1} + \Phi_{a2})^+ - (\Phi_{a1} + \Phi_{a2})^- = 0$. Hence it follows that

$$\Phi_{a2}(z) = -\Phi_{a1}(z) \tag{1.7}$$

For the additional displacements [1, Section 112] we have

$$2\mu_j u'_{aj}(z) = \chi_j\Phi_{aj}(z) + \Phi_{aj}(\bar{z})\frac{d\bar{z}}{dz} - \left(1 - \frac{d\bar{z}}{dz}\right)\overline{\Phi_{aj}(z)} - (z - \bar{z})\overline{\Phi'_{aj}(z)}\frac{d\bar{z}}{dz} \tag{1.8}$$

Consequently, the following equalities hold at the interface $y = 0$ for the derivative in the x direction

$$2\mu_1 u'_{a1}(x) = \chi_1\Phi^+_{a1}(x) + \Phi^-_{a1}(x), \quad 2\mu_2 u'_{a2}(x) = \chi_2\Phi^-_{a2}(x) + \Phi^+_{a2}(x)$$

By (1.7), we can write this equality as

$$2\mu_2 u'_{a2}(x) = -\chi_2\Phi^-_{a1}(x) - \Phi^+_{a1}(x)$$

and the derivative of the difference of displacements is

$$\Delta u'_a(x) = u'_{a1}(x) - u'_{a2}(x) = \left(\frac{\chi_1}{2\mu_1} + \frac{1}{2\mu_2}\right)\Phi^+_{a1}(x) + \left(\frac{\chi_2}{2\mu_2} + \frac{1}{2\mu_1}\right)\Phi^-_{a1}(x)$$

The condition for compensation of the discontinuity $\Delta u'_a(x) = -\Delta u'_f(x)$ leads, by (1.2), to an adjunction problem for the limit values $\Phi^+_{a1}(x)$, $\Phi^-_{a1}(x)$ of the function $\Phi_{a1}(z)$, which is holomorphic off the interface $y = 0$

$$\begin{aligned}&\left(\frac{\chi_1}{2\mu_1} + \frac{1}{2\mu_2}\right)\Phi^+_{a1}(x) + \left(\frac{\chi_2}{2\mu_2} + \frac{1}{2\mu_1}\right)\Phi^-_{a1}(x) = \\ &= \left(\frac{\chi_2}{2\mu_2} - \frac{\chi_1}{2\mu_1}\right)[\Phi_1(x) + \Phi_2(x)] + \left(\frac{1}{2\mu_1} - \frac{1}{2\mu_2}\right)[\Phi_{01}(x) + \Phi_{02}(x)]\end{aligned} \tag{1.9}$$

Since the functions $\Phi_2(z)$, $\Phi_{01}(z)$ are holomorphic in the lower half-plane S_1 and $\Phi_1(z)$, $\Phi_{02}(z)$ are holomorphic in the upper half-plane S_2, the solution of adjunction problem (1.9) is obvious

$$\Phi_{a1}(z) = -\Phi_{a2}(z) = \begin{cases} K_{11}\Phi_{01}(z) + K_{12}\Phi_2(z), & z \in S_1 \\ K_{21}\Phi_1(z) + K_{22}\Phi_{02}(z), & z \in S_2 \end{cases} \tag{1.10}$$

where

$$K_1=\frac{\mu_2-\mu_1}{\mu_1+\chi_1\mu_2},\quad K_{21}=\frac{\chi_2\mu_1-\chi_1\mu_2}{\mu_2+\chi_2\mu_1},\quad K_{12}=\frac{\chi_2\mu_1-\chi_1\mu_2}{\mu_1+\chi_1\mu_2},\quad K_{22}=\frac{\mu_2-\mu_1}{\mu_2+\chi_2\mu_1} \tag{1.11}$$

For half-planes with the same properties ($\mu_1 = \mu_2$, $\chi_1 = \chi_2$) all these coefficients, and accordingly also $\Phi_{a1}(z)$, $\Phi_{a2}(z)$, vanish. If there are no cracks, cuts or inclusions in the upper half-plane, we have $\Phi_2(z) = 0$, $\Phi_{02}(z) = 0$. Then

$$\Phi_{a1}(z)=-\Phi_{a2}(z)=\begin{cases}K_{11}\Phi_{01}(z), & z\in S_1\\ K_{21}\Phi_1(z), & z\in S_2\end{cases}$$

If in addition the interface $y = 0$ is stress-free then, setting $\mu_1 = 0$, we infer from (1.11) that $K_{11} = -1$, $K_{21} = 1$. If the interface is rigidly attached, then, setting $\mu_2 = \infty$, we infer from (1.11) that $K_{11} = 1/\chi_1$, $K_{21} = -\chi_1$.

In all cases, $\Phi_{01}(z)$ and $\Phi_{02}(z)$ are defined by formulae (1.3). The additional stresses $\sigma_{aj}(z)$ and displacements $u_{aj}(z)$ ($j = 1, 2$) are given by formulae (1.6) and (1.8). The solution of the original problem for stresses $\sigma(z)$ and displacements $u(z)$, as mentioned, is given by the sums

$$\sigma(z)=\sigma_{f1}(z)+\sigma_{f2}(z)+\sigma_{aj}(z),\qquad u(z)=u_{f1}(z)+u_{f2}(z)+u_{aj}(z) \tag{1.12}$$

These formulae solve the problem if the functions $\Phi_j(z)$, $\Psi_j(z)$ ($j = 1, 2$) for the arbitrary fictitious stresses $\sigma_f(t)$ are known on a part L_σ of the contours L_1 and L_2 with given $\sigma(t)$ and for the arbitrary fictitious displacements $u_f(t)$ on a part L_u with given displacements $u(t)$. Indeed, using Eqs (1.12) and equating $\sigma(t)$ and $u(t)$ to the given values on L, we obtain equations for the fictitious quantities $\sigma_f(t)$ and $u_f(t)$

$$\begin{aligned}&\sigma_{f1}(t)+\sigma_{f2}(t)+\sigma_{aj}(t)=\sigma(t), \quad t\in L_\sigma\\ &u_{f1}(t)+u_{f2}(t)+u_{aj}(t)=u(t), \quad t\in L_u\end{aligned} \tag{1.13}$$

Note that $\sigma_{aj}(t)$ and $u_{aj}(t)$ depend linearly on $\sigma_{f1}(t)$, $\sigma_{f2}(t)$, $\sigma_{f1}(t)$ and $u_{f2}(t)$. Therefore, Eqs (1.13) are linear in the fictitious quantities. If one is using integral representations of KM potentials with unknown density, then Eqs (1.13) are equations for the density. Once $\sigma_{f1}(t)$, $u_{f1}(t)$ or the density have been found from (1.13) for the contour L_1 in the lower half-plane, a well as $\sigma_{f2}(t)$, $u_{f2}(t)$ or the density for the contour L_2 in the upper half-plane, the functions $\Phi_j(z)$, $\Psi_j(z)$ ($j = 1, 2$) are fully defined. Then formulae (1.4), (1.10), (1.3) and (1.5) determine the KM functions of the original problem.

2. DERIVATION OF BIEs USING INTEGRAL REPRESENTATIONS

Representations of the KM functions by Cauchy and Hadamard integrals provides a convenient tool for deriving complex BIEs in explicit form for the problems under consideration, using the formulae developed above. To abbreviate the notation, we will assume that there are cracks, cuts and inclusions only in the lower of the two coupled half-planes. The subscript j in the notation for the density and the contour will therefore be omitted. It will be retained, however, for the elasticity constants and KM functions, which differ in the lower and upper half-planes. In sufficiently general form, the integral representations of the KM functions may be written as

$$\Phi_1(z)=\frac{1}{2\pi i}\int_L\frac{q'(\tau)d\tau}{\tau-z},\quad \Psi_1(z)=\frac{1}{2\pi i}\int_L\left[\frac{d(Aq)}{\tau-z}-\frac{\bar\tau q'(\tau)}{(\tau-z)^2}d\tau\right] \tag{2.1}$$

The operator A may be chosen by considerations of convenience, depending on the specific properties of the contour L and the boundary conditions on it. In particular, if $Aq = \overline{q(t)}$ we have a representation [14] for a closed contour with given stresses; if $Aq = -\chi_1\overline{q(t)}$, formulae (2.1) correspond to representation with given displacements [15]; if $Aq = \overline{f^+ - f^- - q}$, we obtain the representations underlying the equations for cracks with given stresses on their sides [7]; if $Aq = -2\mu_1(u^+ - u^-) + \chi_1 q$, Eqs (2.1) become the representations underlying the BIEs for cracks with given displacements on the sides [13]. (The last two representations have also been widely used [2, 16].)

Substituting (2.1) into (1.3), and hence into (1.10), we obtain

$$\Phi_{a1}(z) = -\Phi_{a2}(z) = \begin{cases} K_{11}\left[\dfrac{1}{2\pi i}\displaystyle\int_L \overline{q'}\dfrac{\tau-\bar\tau}{(\bar\tau - z)^2}d\bar\tau - \dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{d(\overline{Aq})}{\bar\tau - z}\right], & z\in S_1 \\ K_{21}\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{q'd\tau}{\tau - z}, & z\notin S_1 \end{cases} \tag{2.2}$$

For the additional stresses and derivative of the displacements, using (2.2) in (1.6) and (1.8), we have

$$\sigma_{a1}(z) = \Phi_{a1}(z) + G_{a1}(z), \quad 2\mu_1 u'_{a1}(z) = \chi_1\Phi_{a1}(z) - G_{a1}(z) \tag{2.3}$$

where

$$G_{a1}(z) = K_{11}\left[\frac{1}{2\pi i}\int_L q'\frac{\partial}{\partial z}\frac{z-\bar z}{(\tau-\bar z)^2}(\tau-\bar\tau)d\tau + \frac{1}{2\pi i}\int_L \frac{\partial}{\partial z}\frac{z-\bar z}{\tau - \bar z}d(Aq)\right] -$$
$$-K_{21}\frac{\partial\bar z}{\partial z}\frac{1}{2\pi i}\int_L \frac{q'd\tau}{\tau-\bar z} \tag{2.4}$$

and the quantity $\Phi_{a1}(z)$ is given by the first formula in (2.2).

We stress that the functions $\Phi_{a1}(z)$, $\Psi_{a1}(z)$ and, therefore, $G_{a1}(z)$, $\sigma_{a1}(z)$, $u_{a1}(z)$ are continuous in the lower half-plane. This important property enables us easily to extend known BIEs to problems involving coupled half-planes. Indeed, by (1.4), in the case in question, the KM functions in the lower half-plane are $\Phi(z) = \Phi_1(z) + \Phi_{a1}(z)$, $\Psi(z) = \Psi_1(z) + \Psi_{a1}(z)$. Substituting them into the boundary conditions for the stresses and the derivative of the displacements on L we obtain

$$\Phi_1^{\pm}(t) + \overline{\Phi_1^{\pm}(t)} + t[\overline{\Phi_1'^{\pm}(t)} + \overline{\Psi_1^{\pm}(t)}]\frac{d\bar t}{dt} = \sigma^{\pm} - \sigma_{a1} \tag{2.5}$$

$$\chi_1\Phi_1^{\pm}(t) - \overline{\Phi_1^{\pm}(t)} - t[\overline{\Phi_1'^{\pm}(t)} + \overline{\Psi_1^{\pm}(t)}]\frac{d\bar t}{dt} = 2\mu_1[u'^{\pm} - u'_{a1}(t)] \tag{2.6}$$

where $\sigma_{a1}(t)$, $u_{a1}(t)$ are the additional stresses and derivatives of displacements, defined by formulae (2.3). As mentioned, they are continuous across L and are therefore used without superscripts $\pm$. A superscript plus (minus) denotes limit values to the left (right) of the direction of motion along the contour.

It follows from (2.5) and (2.6) that the complex BIEs corresponding to singular solutions for the whole plane may be extended to problems involving coupled half-planes if one replaces $\sigma^{\pm}$ by $\sigma^{\pm} - \sigma_{a1}$ and $u^{\pm}$ by $u^{\pm} - u_{a1}$, where as before $\sigma^{\pm}$ and $u^{\pm}$ and the true stresses and displacements on the contour. When that is done, the discontinuities of the stresses $\Delta\sigma = \sigma^{\pm} - \sigma^{-}$ and of the displacements $\Delta u = u^{+} - u^{-}$ on the contour remain unchanged, since the additional fields are continuous. The density q and its derivative q' are also given by the same expressions in terms of the discontinuities of the true displacements and stresses

$$q(t) = \frac{2\mu_1\Delta u + \Delta f}{\chi_1 + 1}, \quad q'(t) = \frac{2\mu_1\Delta u' + \Delta\sigma}{\chi_1 + 1}$$

where, as before, the subscript 1 indicates that the elasticity parameter refers to the lower half-plane. Finally, one can write down complex equations for problems with coupled half-planes by applying these transformations to the BIEs for an infinite plane. Rather than present all the equations, we will confine ourselves to three illustrations, using notation which, for brevity, will be used in the next section too

$$m_1 = \frac{1}{2\pi(\chi_1+1)}, \quad S(\tau,t) = \frac{1}{\tau - t}, \quad H(\tau,t) = \frac{1}{(\tau-t)^2}, \quad R(\tau,t) = \frac{\tau-\bar\tau}{\tau - t}, \quad Q(\tau,t) = \frac{\tau-\bar\tau}{(\bar\tau - t)^2}$$

The singular equation developed in [7] for a plane with cuts takes the following form after the transformation just described for coupled half-planes

$$-2\mu_1 i m_1\int_L\left[2\Delta u' S(\tau,t) + \Delta u'\frac{\partial k_1}{\partial t}d\tau + \overline{\Delta u'}\frac{\partial k_2}{\partial t}d\bar\tau\right] + B\Delta\sigma + \sigma_{a1} = \frac{\sigma^{+}+\sigma^{-}}{2}, \quad t\in L \tag{2.7}$$

where

$$B\Delta\sigma = -im_1 \int_L \left[(1-\chi_1)\Delta\sigma S(\tau,t) - \chi_1 \Delta\sigma \frac{\partial k_1}{\partial t} d\tau + \overline{\Delta\sigma} \frac{\partial k_2}{\partial t} d\bar{\tau}\right]$$

and in this case it follows from (2.3) and (2.4) that

$$\sigma_{a1} = -2\mu_1 i m_1 \int_L \left[\Delta u' \frac{\partial k_3}{\partial t} d\tau + \overline{\Delta u'} \frac{\partial k_4}{\partial t} d\bar{\tau}\right] + B_{a1}\Delta\sigma \tag{2.8}$$

$$B_{a1}\Delta\sigma = -im_1 \int_L \left\{\Delta\sigma \frac{\partial}{\partial t}[k_3 + (\chi_1+1)K_{11}\ln(\bar{\tau}-t)]d\tau + \overline{\Delta\sigma}\frac{\partial}{\partial t}[k_4 - (\chi_1+1)K_{11}R(t,\tau)]d\bar{t}\right\}$$

$$k_1(\tau,t) = \ln\frac{\tau-t}{\bar{\tau}-\bar{t}}, \quad k_2(\tau,t) = \frac{\tau-t}{\bar{\tau}-\bar{t}} \tag{2.9}$$

$$k_3(\tau,t) = -K_{11}[\ln(\bar{\tau}-t) + (t-\bar{t})Q(\bar{\tau},\bar{t})] + K_{21}\ln(\tau-\bar{t}), \quad k_4(\tau,t) = K_{11}[R(t,\tau)+R(\tau,t)]$$

The hyper-singular equation is obtained from (2.7) by integrating the terms containing the derivative of the discontinuity of the displacements by parts

$$-2\mu_1 i m_1 \int_L \left[2\Delta u H(\tau,t) - \Delta u \frac{\partial}{\partial t} dk_1 - \overline{\Delta u}\frac{\partial}{\partial t} dk_2\right] + B\Delta\sigma + \sigma_{a1} = \frac{\sigma^+ + \sigma^-}{2}, \quad t\in L \tag{2.10}$$

$$\sigma_{a1} = -2\mu_1 i m_1 \int_L \left[-\Delta u \frac{\partial}{\partial t} dk_3 - \overline{\Delta u}\frac{\partial}{\partial t} dk_4\right] + B_{a1}\Delta\sigma$$

(the term σ_{a1} is obtained from (2.8) by analogous integration by parts). The differentials in (2.10) and in subsequent formulae are evaluated with respect to τ.

For inclusions and systems of blocks (granules) in the lower half-plane, representations (2.1) are conveniently used on the assumption that the stresses on the boundaries are continuous. Then the operator A occurring in (2.1) is defined by the formula $Aq = -q(t)$, and consequently $dAq = -q'(\tau)d\tau$. Then, using these representations for the matrix, the inclusions and the blocks (granules), we have an extension of the equation of [8] to block systems. We will write it in a hyper-singular form which extends the equation obtained in [9] to coupled half-planes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_L \left\{2\Delta u H(\tau,t)d\tau - \Delta u \frac{\partial}{\partial t} d(k_1+k_3) - \Delta\bar{u}\frac{\partial}{\partial t} d(k_2+k_4) - \right.\\
&-(a_3 - 2a_1)\sigma S(\tau,t)d\tau + \sigma\frac{\partial}{\partial t}[(a_1-a_3)k_1 + a_1 k_3 + a_3 K_{11}\ln(\bar{\tau}-t)]d\tau + \\
&\left.+\bar{\sigma}\frac{\partial}{\partial t}[a_1 k_2 + a_1 k_4 - a_3 K_{11} R(t,\tau)]d\bar{\tau}\right\} = \frac{1}{2} a_2 \sigma(t), \quad t\in L
\end{aligned} \tag{2.11}$$

$$a_1 = \frac{\chi^+ + 1}{2\mu^+} + \frac{\chi^- + 1}{2\mu^-}, \quad a_2 = \frac{\chi^+ + 1}{2\mu^+} + \frac{\chi^- + 1}{2\mu^-}, \quad a_3 = \frac{\chi^+ + 1}{2\mu^+} - \frac{\chi^- + 1}{2\mu^-}$$

The kernels k_1, k_2, k_3 and k_4 are given by formulae (2.9).

The contour L may include cracks and holes; if the hole is circumscribed in the clockwise sense, we must assume that $1/\mu^- = 0$, $u^- = 0$. If the contour L consists only of cracks subject to stresses of equal value but opposite in direction, we have $a_1 = 0$, $a_3 = 0$, $a_2 = 2(\chi_1 + 1)/(2\mu_1)$. Then Eq. (2.11) is converted into Eq. (2.10), where in this case $\Delta\sigma = 0$ ($\sigma^+ = \sigma^- = \sigma$). In the special case of a single load-free half-plane, Eq. (2.11) will be identical with an equation obtained previously [5] by other means—using complex singular solutions. Such solutions for coupled half-planes will be discussed in the following section.

3. SINGULAR SOLUTIONS FOR COUPLED HALF-PLANES

Another approach to the derivation of complex BIEs is to follow the method of real potentials. This approach [6, 10] uses complex forms of singular solutions, that is, solutions for point forces. Such solutions exist for a whole plane (see, e.g. [1]), for a half-plane with a free boundary [11] and for a half-plane with a free or rigidly attached boundary [12]. In what follows, the formulae of Section 1 will be used to derive complex singular solutions corresponding, first, to isolated forces and, second, to periodic systems of forces. The first serve for non-periodic problems and the second for periodic problems involving coupled half-planes.

Isolated force. Suppose point unit force is applied at a point τ of the lower half-plane in the direction of the x (or y) axis. Quantities corresponding to this case will be labelled by a superscript x (or y). There are known formulae for the KM functions when the force is applied to a whole homogeneous plane with the properties of the lower half-plane [1]. Using these functions and the formulae of Section 1, one arrives after some reduction at expressions for the additional functions.

For the KM functions, we have

in the lower half-plane

$$
\begin{aligned}
\varphi_{a1}^{x}(z,\tau) &= -m_1 K_{11}[-\chi_1 \ln(z-\bar{\tau}) + R(\tau,z)] \\
\psi_{a1}^{x}(z,\tau) &= -m_1\{-K_{21}\ln(z-\bar{\tau}) + K_{11}z[\chi_1 S(z,\bar{\tau}) - Q(\tau,z)]\} \\
\varphi_{a1}^{y}(z,\tau) &= -i\,m_1 K_{11}[-\chi_1 \ln(z,\bar{\tau}) - R(\tau,z)] \\
\psi_{a1}^{y}(z,\tau) &= -i\,m_1\{K_{21}\ln(z-\bar{\tau}) + K_{11}z[\chi_1 S(z,\bar{\tau}) + Q(\tau,z)]\}
\end{aligned}
$$

in the upper half-plane

$$
\begin{aligned}
\varphi_{a2}^{x}(z,\tau) &= m_1 K_{21}\ln(z-\tau) \\
\psi_{a2}^{x}(z,\tau) &= m_1\{-K_{21}zS(z,\tau) + K_{11}[\chi_1\ln(z-\tau) - R(\bar{\tau},z)]\} \\
\varphi_{a2}^{y}(z,\tau) &= i\,m_1 K_{21}\ln(z-\tau) \\
\psi_{2}^{y}(z,\tau) &= i\,m_1\{-K_{21}zS(z,\tau) - K_{11}[\chi_1\ln(z-\tau) + R(\bar{\tau},z)]\}
\end{aligned}
$$

The additional displacements, principal vector of forces and stresses are obtained by substituting these formulae into the KM formulae. Since the resulting formulae are cumbersome, we will present them explicitly only for the key quantities—the additional displacements. These are given, after the above-mentioned substitution, by the following formulae

in the lower half-plane

$$
\begin{aligned}
2\mu_1 u_{a1}^{x}(z,\tau) &= -m_1\{K_{21}\ln(\bar{z}-\tau) + K_{11}[-\chi_1^2\ln(z-\bar{\tau}) + \chi_1 R(\tau,z) + \chi_1 R(z,\tau) + R(z,\tau)R(\bar{\tau},\bar{z})]\} \\
2\mu_1 u_{a1}^{y}(z,\tau) &= -i\,m_1\{K_{21}\ln(\bar{z}-\tau) + K_{11}[-\chi_1^2\ln(z-\bar{\tau}) - \chi_1 R(\tau,z) - \chi_1 R(z,\tau) + \\
&\quad + R(z,\tau)R(\bar{\tau},\bar{z})]\}
\end{aligned}
$$

in the upper half-plane

$$
\begin{aligned}
2\mu_2 u_{a2}^{x}(z,\tau) &= -m_1\{-K_{21}[\chi_2\ln(z-\tau) - R(z,\bar{\tau})] + K_{11}[\chi_1\ln(\bar{z}-\bar{\tau}) - R(\tau,\bar{z})]\} \\
2\mu_2 u_{a2}^{y}(z,\tau) &= -i\,m_1\{-K_{21}[\chi_2\ln(z-\tau) + R(z,\bar{\tau})] + K_{11}[\chi_1\ln(\bar{z}-\bar{\tau}) + R(\tau,\bar{z})]\}
\end{aligned}
$$

The fundamental solutions for coupled half-planes are the sums of fundamental solutions for a whole plane and the additional quantities found: $u^x(z,\tau) = u_{lj}^x(z,\tau) + u_{aj}^x(z,\tau)$, $u^y(z,\tau) = u_{1j}^y(z,\tau) + u_{aj}^y(z,\tau)$ $(j = 1, 2)$ where $u_{ij}^x(z,\tau)$ are complex fundamental solutions for the displacement field in the case of a whole plane

$$
\begin{aligned}
2\mu_j u_{1j}^{x}(z,\tau) &= -m_1[\chi_j\ln(z-\tau) + \chi_1\ln(\bar{z}-\bar{\tau}) - R(z,\tau) - R(\tau,z)] \\
2\mu_j u_{1j}^{y}(z,\tau) &= -i\,m_1[\chi_j\ln(z-\tau) + \chi_1\ln(\bar{z}-\bar{\tau}) + R(z,\tau) + R(\tau,z)]
\end{aligned}
$$

In all these formulae the coefficients K_{11} and K_{12} are defined by formulae (1.11). In the special case of one load-free half-plane we have $K_{11} = -1$, $K_{21} = 1$, and the results are equivalent to previously known formulae [11] (presented in explicit form in [10]). In the case of a rigidly attached half-plane, when $K_{11} = 1/\chi$, $K_{21} = -\chi_1$, and in the general case of two coupled half-planes, these formulae are still unpublished, as far as we know.

Periodic system of forces. In exactly the same way, one obtains fundamental solutions for a π-periodic system of forces applied at the points $\tau = m\pi$ of the lower half-plane ($m = \ldots, -2, -1, 0, 1, 2, \ldots$). (For an arbitrary real period a it will suffice to add the factor π/a before the variables z and τ—and hence also before $\bar{z}$ and $\bar{\tau}$—in all subsequent formulae.) One begins with the formulae for a whole plane, given the same disposition of periodic systems of forces acting in the direction of the x or y axis [10]. Using these formulae according to the scheme of Section 1, one obtains expressions for the KM functions determining the total stresses. The KM functions defining the displacements and principal vector of forces are obtained by integrating these expressions; after substituting the result into the KM formulae for displacements one obtains expressions for the required singular solutions:
in the lower half-plane

$$\begin{aligned}
2\mu_1 u^x(z,\tau) = -m_1\{&\chi_1 \ln\sin(z-\tau) + \chi_1 \overline{\ln\sin(z-\tau)} - [(z-\bar{z}) - (\tau-\bar{\tau})]\operatorname{ctg}(\bar{z}-\bar{\tau}) + \\
&+K_{21}\overline{\ln\sin(z-\bar{\tau})} + K_{11}[-\chi_1^2 \ln\sin(z-\bar{\tau}) - \chi_1(\tau-\bar{\tau})\operatorname{ctg}(z-\bar{\tau}) + (z-\bar{z})(\chi_1 \operatorname{ctg}(\bar{z}-\tau) + \\
&+(\tau-\bar{\tau})\operatorname{cosec}^2(\bar{z}-\tau))]\} \\
2\mu_1 u^y(z,\tau) = -i\, m_1\{&\chi_1 \ln\sin(z-\tau) + \chi_1 \overline{\ln\sin(z-\tau)} + [(z-\bar{z}) - (\tau-\bar{\tau})]\operatorname{ctg}(\bar{z}-\bar{\tau}) + \\
&+K_{21}\overline{\ln\sin(z-\bar{\tau})} + K_{11}[-\chi_1^2 \ln\sin(z-\bar{\tau}) + \chi_1(\tau-\bar{\tau})\operatorname{ctg}(z-\bar{\tau}) + (z-\bar{z})(-\chi_1 \operatorname{ctg}(\bar{z}-\tau) + \\
&+(\tau-\bar{\tau})\operatorname{cosec}^2(\bar{z}-\tau))]\}
\end{aligned}$$

in the upper half-plane

$$\begin{aligned}
2\mu_2 u^x(z,\tau) = -m_1\{&\chi_2 \ln\sin(z-\tau) + \chi_1 \overline{\ln\sin(z-\tau)} - [(z-\bar{z}) - (\tau-\bar{\tau})]\overline{\operatorname{ctg}(z-\tau)} - \\
&-K_{21}[\chi_2 \ln\sin(z-\tau) - (z-\bar{z})\overline{\operatorname{ctg}(z-\tau)}] + K_{11}[\chi_1 \overline{\ln\sin(z-\tau)} + (\tau-\bar{\tau})\overline{\operatorname{ctg}(z-\tau)}]\} \\
2\mu_2 u^y(z,\tau) = -i\, m_1\{&\chi_2 \ln\sin(z-\tau) + \chi_1 \overline{\ln\sin(z-\tau)} + [(z-\bar{z}) - (\tau-\bar{\tau})]\overline{\operatorname{ctg}(z-\tau)} - \\
&-K_{21}[\chi_2 \ln\sin(z-\tau) + (z-\bar{z})\overline{\operatorname{ctg}(z-\tau)}] + K_{11}[\chi_1 \overline{\ln\sin(z-\tau)} - (\tau-\bar{\tau})\operatorname{ctg}(z-\tau)]\}
\end{aligned}$$

In the special cases of a lower half-plane with free boundary ($K_{11} = -1$, $K_{21} = 1$) and with rigidly attached boundary ($K_{11} = 1/\chi_1$, $K_{21} = -\chi_1$), one obtains fundamental solutions identical with previous results [16], apart from terms corresponding to the homogeneous state of the half-plane.

Analysis shows that the complex displacements $u^x(z, \tau)$ defined by these formulae have a cyclic constant iA in both half-planes; the displacements $u^y(z, \tau)$ have a cyclic constant $-A$, where

$$A = (K_{21} + \chi_1^2 K_{11})/[4\mu_1(\chi_1 + 1)] = [\chi_2(1 - K_{21}) - \chi_1(K_{11} + 1)]/[4\mu_2(\chi_1 + 1)]$$

(the last part of the equality is obtained by using relations (1.11)).

Note that all the formulae for a periodic system of forces may be obtained from the corresponding formulae for isolated forces by formally replacing $\ln\zeta$ by $\ln \sin \zeta$, $1/\zeta$ by $\operatorname{ctg} \zeta$ and $1/\zeta^2$ by $1/\sin^2 \zeta$.

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